BAYESIAN DECISION MAKING AND E-LEARNING: HOW TO COMBINE?

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Abstract: The paper describes an approach to the formulation of the decision-making tasks via specification including such specifying constituents as system, experience, ignorance, admissible strategy, loss function and decision. The paper shows how to construct the offered specification on two examples of decision-making problems. One of them is a simple example from the everyday life; as the second example the fully probabilistic control design in its state-space setting has been taken. The paper shows how this way of the material organizing can be applied as the educational resources of the e-learning software.

Keywords: Decision-making, Fully probabilistic design, E-learning

1 INTRODUCTION

Nowadays such an approach to the education as *e-learning* comes into our life with more and more force, becoming a popular one in most of universities and educational institutions. But it should be noted that such an important area of e-learning software content as math-oriented material (particularly, Bayesian decision making) are still not sufficiently supported nowadays. It can be said both in terms of a conceptual approach to a structure of the information presented and from the point of view of implementation of the decision-making (DM) tasks in math-oriented educational software tools. The paper tries to reduce the lack of the solutions in this field and offers its own one.

The basic tasks of Bayesian decision making, a lot of books and articles on which are available, see (Jensen, 2001; Berger, 1985; Bernardo and Smith, 1997), and which is studied by many Ph. D. students, can be conditionally divided among estimation (point and set), testing of hypotheses, prediction and control. The new approach to organize the educational math-oriented text is proposed:

to formulate the decision-making tasks uniformly with the help of the specification.

The specification proposed consists of six following constituents: *system, experience, ignorance, admissible strategy, loss function* and *decision*. These constituents are, in essence, the concepts, which a decision maker should define when formulating a task. They are not always so obvious as it may seem.

In order to determine them a decision maker should know the definitions of the concepts (Kárný *et al.*, 2005):

- *System* is a part of the world that is of interest for a decision maker who should either describe or influence it.
- *Experience* is knowledge about the system, available to the decision maker for the selection of the particular decision.
- *Ignorance* concerns knowledge about the system, unavailable to the decision maker.
- *Admissible strategy* is a mapping from experience to actions, that meets physical constraints, implied by the system and the actuator.
- *Loss function* quantifies the degree of achievement of the DM aim by assigning to each realization of behavior a number on real line.
- *Decision* is the value of a quantity that can be directly chosen by the decision maker for reaching decision maker's aims.

The paper will show how the DM tasks at different levels – both the simple task from everyday life and the complex theoretical DM problem – can be structured according to the specification.

2 DECISION-MAKING TASK SPECIFICATION APPLICATION

2.1 Example from everyday life

In order to demonstrate a specification of the DM task, let's make a simple example from everyday life. Let a decision maker be an usual student. Assume that a student has the following *aim*: to become a good specialist, who has a good knowledge and excellent diploma. Such the aim can be expressed in another way as it follows.

Aim: to minimize a divergence between the knowledge of the student and knowledge of a good specialist.

For this example the specification can be presented in the following way.

- *System:* is the student, who has the lectures, study material and teachers as the inputs and exams and marks (in principle, the student's knowledge) as the outputs;
- *Experience:* past marks, mapping a student's level of knowledge; student's abilities;
- *Ignorance:* such factor as the exam, which can include the psychological state of student's health, a teacher's behavior, transport problems;
- *Admissible strategy:* time and energy of the student that can be devoted to study, restricted by schedule, number of exams and subjects, other duties;
- *Loss function:* the quality of knowledge expressed in marks; time spent to study;

- *Decision:* how much time and energy the student will devote to study, taking into account all the restrictions.

This example from life is very simple and, probably, it does not stand up to criticism. But the principle of composing the specification is clear now.

2.2 *Fully probabilistic design*

The fully probabilistic design (FPD) (Kárný, 1996), which can be interpreted as a superstructure of Bayesian DM on the whole and under the corresponding setting can include majority of DM tasks, is a good example to demonstrate the approach.

Preliminaries The next notations are used throughout this section of the paper: $\equiv$ is equality by definition; X^* denotes a set of X-values; $\mathring{X}$ means cardinality of a finite set X^*; $f(\cdot|\cdot)$ denotes probability density function (pdf) that is assumed to exist; t labels discrete-time moments, $t \in t^* \equiv \{1, \ldots, \mathring{t}\}$; $\mathring{t} < \infty$ is a given control horizon; $d_t = (y_t, u_t)$ is the data record at time t consisting of an observed system output y_t and of an optional system input u_t; x_t is an unobserved system state; d, x are assumed to be finite-dimensional; $X(t)$ denotes the sequence $(X_1, \ldots, X_t)$, $X(t) \in \{y(t), u(t), x(t)\}$.

The FPD uses the Kullback-Leibler (KL) divergence (Kullback and Leibler, 1951)

$$\mathcal{D}\left(f||\tilde{f}\right) \equiv \int f(X) \ln \left(\frac{f(X)}{\tilde{f}(X)}\right) dX \tag{1}$$

to measure the proximity of a pair of pdfs $f, \tilde{f}$ acting on a set X^*. The key property of KL divergence is

$$\mathcal{D}(f||\tilde{f}) \geq 0,\ \mathcal{D}(f||\tilde{f}) = 0 \text{ iff } f = \tilde{f} \text{ almost everywhere on } X^*. \tag{2}$$

The FPD deals with the controlled closed loop system and describes it entirely in probabilistic terms. In order to represent the desired joint distribution of the considered closed loop variables the FPD requires the *ideal pdf* to be introduced.

The aim of the FPD task can be formulated in the following way:

Find admissible control strategy minimizing the KL divergence $\mathcal{D}\left(f_{\mathring{t}}||\,{}^{I}f_{\mathring{t}}\right)$,

where $f_{\mathring{t}}$ is the joint pdf describing the controlled closed loop and ${}^{I}f_{\mathring{t}}$ is the ideal pdf.

Despite the aim is formulated, it remains rather indistinct without precise specification of DM task.

FPD specification The task of FPD can be specified with the help of constituents proposed in the following way.

System: The joint pdf describing the controlled closed loop can be decomposed with the help of the chain rule (Peterka, 1981).

$$f(d(\mathring{t}), x(\mathring{t})|x_0) = \tag{3}$$

$$= f(x_0) \prod_{t \in t^*} f(y_t|u_t, x(t), d(t-1)) f(x_t|u_t, x(t-1), d(t-1)) f(u_t|x(t-1), d(t-1)).$$

From this relation it follows that the system is described by *observation model* $f(y_t|u_t, x(t), d(t-1))$, the *state evolution model* $f(x_t|u_t, x(t-1), d(t-1))$ and the general *randomized controller* $f(u_t|x(t-1), d(t-1))$ (Kárný and Guy, 2004).

The following *assumptions* are made about the system models:

- the unobserved system state x_t does not depend on the past history of the system, i.e.

$$f(x_t|u_t, x(t-1), d(t-1)) = f(x_t|u_t, x_{t-1});$$

- the probability distribution of the observed system output y_t is determined only by the current system input u_t and the system state x_t, i.e.

$$f(y_t|u_t, x(t), d(t-1)) = f(y_t|u_t, x_t).$$

Admissible strategies: The set of admissible actions is restricted by the following *assumption*:

- admissible control strategies generate the system input u_t from the observed data history $d(t-1)$ and ignore the unobserved states $x(t-1)$, i.e.

$$f(u_t|x(t-1), d(t-1)) = f(u_t|d(t-1)).$$

Hence, the joint pdf (3) reduces to

$$f(d(\mathring{t}), x(\mathring{t})|x_0) f(x_0) = \prod_{t \in t^*} f(y_t|u_t, x_t) f(x_t|u_t, x_{t-1}) f(u_t|d(t-1)) f(x_0), \tag{4}$$

and the models of the system define the conditional probability distributions

$$f(y_t|u_t, x_t),\ f(x_t|u_t, x_{t-1}),\ f(u_t|d(t-1)).$$

Loss function: The loss function for the FPD task follows from its DM aim, namely, from searching the admissible control strategy minimizing the KL divergence (1). It means, the KL divergence between the joint pdf described the closed control loop and the decision maker's ideal pdf will be the loss function.

The decision maker's ideal pdf is constructed in the way analogous to (4)(note an appearance of superscript ^{I})

$${}^I f(d(\mathring{t}), x(\mathring{t})|x_0)\, {}^I f(x_0) = \prod_{t \in t^*} {}^I f(y_t|u_t, x_t)\, {}^I f(x_t|u_t, x_{t-1})\, {}^I f(u_t|d(t-1)) f(x_0), \tag{5}$$

where pdf ${}^I f(y_t|u_t, x_t)$ describes the ideal model of observation, ${}^I f(x_t|u_t, x_{t-1})$ – the ideal model of state evolution, ${}^I f(u_t|d(t-1))$ – the ideal control law.

It is obviously that the prior pdf on initial states x_0^* cannot be influenced by the optimized control strategy, so ${}^I f(x_0) = f(x_0)$.

Experience: The knowledge about the system, available to the decision maker for achieving the aim, includes the past outputs, the model structure and model parameters. Theoretically, the problem of identification could be included in a list of the tasks for FPD, but here the parameters are supposed to be known.

Ignorance: Such knowledge about the system as future inputs, future outputs and all the system states is not available to the decision maker.

Decision: It has been already mentioned that the FPD is considered in its space-state setting. According to the previous constituent, the knowledge about the system states is unavailable to achieve the DM aim. Therefore, the decision maker should solve the state estimation task.

Bayesian filtering in closed control loop includes the next steps (Peterka, 1981) on conditions that the prior pdf $f(x_0)$ is available:

$$f(x_t|u_t, d(t-1)) = \int f(x_t|u_t, x_{t-1}) f(x_{t-1}|d(t-1))\, dx_{t-1} \tag{6}$$

$$f(x_t|d(t)) = \frac{f(y_t|u_t, x_t) f(x_t|u_t, d(t-1))}{\underbrace{\int f(y_t|u_t, x_t) f(x_t|u_t, d(t-1))\, dx_t}_{f(y_t|u_t, d(t-1))}}, \tag{7}$$

where (6) is understood as the time updating, while relation (7) as data updating.

Thus the state estimation task built in the FPD is solved. The successive task, necessary to achieve the decision making aim, is the regulation problem.

Let the joint pdf $f(x(\mathring{t}), d(\mathring{t})|x_0)$ and the ideal one ${}^I f(x(\mathring{t}), d(\mathring{t})|x_0)$ be considered according to the previous constituents of the specification.

The optimal admissible control strategy, which minimizes $\mathcal{D}\left(f_{\mathring{t}}||\, {}^I f_{\mathring{t}}\right)$ is randomized one given by the pdfs

$${}^o f(u_t|d(t-1)) = {}^I f(u_t|d(t-1)) \frac{\exp[-\omega(u_t, d(t-1))]}{\gamma(d(t-1))}, \quad t \in t^*, \tag{8}$$

$$\gamma(d(t-1)) \equiv \int {}^I f(u_t|d(t-1)) \exp[-\omega(u_t, d(t-1))]\, du_t.$$

Starting with $\gamma(d(\mathring{t})) \equiv 1$, the functions $\omega(u_t, d(t-1))$ are generated recursively for $t = \mathring{t}, \mathring{t}-1, \ldots, 1$ in a backward manner, as it is shown below

$$\omega(u_t, d(t-1)) \equiv \int \Omega(u_t, d(t-1), x_{t-1}) f(x_{t-1}|d(t-1)) dx_{t-1}, \tag{9}$$

where $f(x_t|d(t))$ is updated according to the filtering operations (6)-(7) and

$$\Omega(u_t, d(t-1), x_{t-1}) \equiv \tag{10}$$

$$\equiv \int f(y_t|u_t, x_t) f(x_t|u_t, x_{t-1}) \ln \left(\frac{f(y_t|u_t, x_t) f(x_t|u_t, x_{t-1})}{\gamma(d(t))\, {}^I f(y_t|u_t, x_t)\, {}^I f(x_t|u_t, x_{t-1})} \right) dy_t\, dx_t.$$

The result is based on the use of the definition of the KL divergence (1), its basic property (2), Fubini theorem on multiple integration (Rao, 1987), marginalization and normalization of pdfs and the chain rule for them (Peterka, 1981).

3 CONCLUSIONS

Thus, with the help of specifying the DM tasks in the described way, it is possible to create a great number of the individual case studies, each with its own specification. Among the advantages of the approach it is worth enumerating the next ones: (i) uniformly specified examples give maximum information about task's features to the students; (ii) with a knowledge of specification the students can compare the different algorithms and are prepared to see a task implementation in MATLAB; (iii) terminological connection of particular tasks with general theory gives the students better understanding of the subject. A collection of specified examples and theory are available at moodle.utia.cas.cz.

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